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Transition from stationary to traveling localized patterns in a two-dimensional reaction-diffusion system

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The bifurcation from localized stationary solutions to traveling patterns is studied in two-dimensional reaction-diffusion systems. In the case of local kinetics, which is of the Bonhoeffer–van der Pol type, the bifurcation point can be computed directly in terms of the stationary solution for arbitrary parameters. A stripelike pattern, which is the extension of a one-dimensional localized pattern into the second spatial dimension, is considered as a concrete example of such a pattern. The branching mode is computed for piecewise linear kinetics in a singular limit. The bifurcation is predominantly subcritical. Eventually the linear stability analysis of the branching traveling pattern is performed. Following the bifurcating branch, the last mode to become stable is a laterally wavy perturbation with an arbitrarily long wavelength.

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I. INTRODUCTION

In recent times considerable interest has arisen in solitary stationary patterns in dissipative media, called dissipative solitons [1]. These occur quite naturally in several experimental setups of the reaction-diffusion type. As examples of spatially one-dimensional systems, patterns of current density occur in gas discharge systems [2], in semiconductor devices [3], and on coupled electrical oscillators [4]. Further, temperature patterns have been observed on catalytic ribbons [5]. There has also been a lot of work done in two-dimensional systems. In addition to several studies on semiconductor devices [6], we want to mention an ac-gas discharge system [7], which nicely allows direct observation by eye. An appropriate phenomenological description of such systems is given by an activator inhibitor model with Bonhoeffer–van der Pol kinetics [8],

$$\begin{aligned} \tau\varepsilon\dot{v} &= \varepsilon^2\Delta v + f(v,a) - w, \\ \dot{w} &= \Delta w + v - \gamma w, \quad \gamma > 0, \end{aligned} \tag{1}$$

with $f(v,a)$ being a cubiclike nonlinearity as in Fig. 1(a) depending on some parameter a. For suitable parameters, the occurrence of dissipative solitons (DSs) could be demonstrated numerically [8]. However, the complex structure of (1) obstructs a detailed analytical understanding of such patterns in the general case. Nevertheless, the approach of the singular limit in parameter space $\varepsilon \to 0$ leads to sharp variations of v and thus gives way to approximating analytical results for strongly separated scales [9,10]. The DS can be interpreted as an excited domain with v slaved to the excited branch of the characteristic $v_+(w)$; compare Fig. 1. Outside this region, v is located on the lower branch of the characteristic $v_-(w)$. The dynamics within the domains is thus reduced to a one component system. The different domains are connected by sharp frontlike boundaries. At

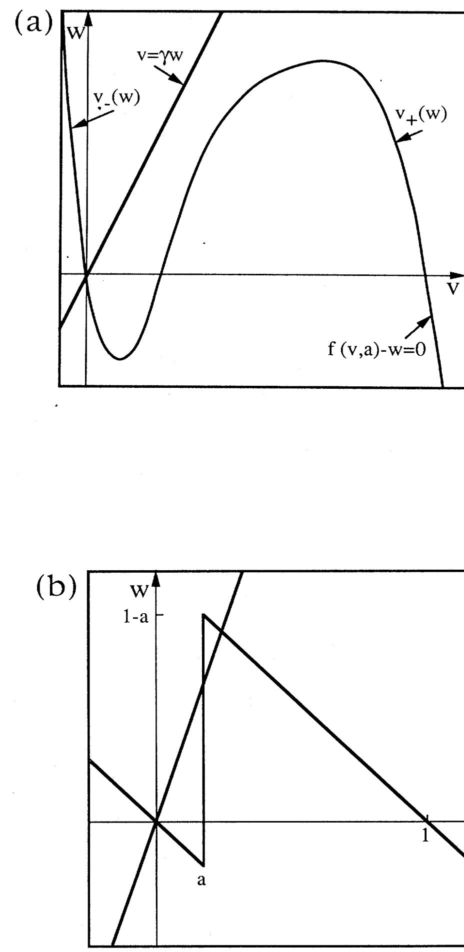

FIG. 1. (a) Typical nullclines of Eq. (1). For most of the article we study the piecewise linear caricature presented in (b). Our analysis does not depend on the fact that there is a single fixed point of the kinetics.

these boundaries the variable v changes inside a layer of width $O(\varepsilon)$ from the excited to the lower branch of $f(v,a)$. Although the reaction term $v-\gamma w$ rises steeply within this range, the large diffusion of w allows us to consider w as a constant there compared to the variation of v. In this way, the translation dynamics of the fronts can be reduced to a free boundary problem [11]. Their velocities are exclusively determined by the local inhibitor value w at the position p of the respective front. This value $w(p)$ itself is determined by the spatiotemporal dynamics of the domain variable w, closing the feedback loop in our explanation. For a front to be stationary in this lowest order approximation, $w(p)$ must fulfill a Maxwell condition $\int_{v_-(w(p))}^{v_+(w(p))}[f(v,a)-w(p)]\,dv=0$. Typical DSs for different values of ε are presented in Fig. 2. In order to obtain explicit analytical results, even for $\varepsilon>0$, quite often a somewhat artificial characteristic is considered [12,13], which is piecewise linear [Fig. 1(b)]:

$$f(v,a)=H(v-a)-v. \qquad (2)$$

H is the Heaviside step function and a satisfies $0<a<1$. DSs of the form given in Fig. 2 exist in the parameter range $\frac{\gamma}{2(1+\gamma)}<a<\frac{1}{2}$. Convergence toward the limiting value $a=\frac{1}{2}$ lets the DS collapse, whereas the approach of the lower limit leads to a diverging width of the DS. In general, the stability of a DS depends on the relative time scales of front movement and domain dynamics of w. For a fixed front position the domain dynamics is stable, whereas for fixed domains the front dynamics is unstable. Decreasing τ leads to an acceleration of front dynamics and, as a consequence, to a destabilization of the DS. In a spatially infinite system with rather general reaction kinetics, two fundamentally different destabilizing modes have analytically been found close to the singular limit: breathing [14] and traveling [15,16]. Consideration of the piecewise linear characteristic again explicitly renders patterns and bifurcation points [13,17]. Breathing DSs can be found experimentally in p-type germanium semiconductor devices [18]. Traveling patterns are well known from the study of excitable media; see [19], e.g. The only difference from (1) is the fact that the diffusion of w is either weak or nonexistent there. This excludes the existence of stable stationary DSs. Nevertheless, a bifurcation cascade has been found that leads from the homogeneous equilibrium state to traveling solitary patterns [20].

In an experimental situation, purely traveling patterns can only appear in a circular systems. Due to the mirror symmetry of the stationary DS, the bifurcation to traveling patterns is a pitchfork in the translationally symmetric system. Boundaries, for example with zero flux conditions, break the translational symmetry of the system and obstruct the traveling mode. They consequently lead to a perturbation of the pitchfork bifurcation. The result is a Hopf bifurcation due to the repulsive boundaries. The branching mode, then, is a moving DS, which is reflected when approaching a boundary, i.e., swinging dynamics results. Exactly this could be found in experiment [21–23]. We will come back to that point later in this paper in a slightly different context. The reflection at the Neumann boundary is an essential difference in comparison with excitable media. Without or with weak diffusion of w the pulse is destroyed instead of being reflected.

Let us now consider the situation in a two-dimensional system. There are two different elementary forms of stationary DSs with high symmetry. One is a radially symmetrical spot. The other is the extension of the one-dimensional pattern into the additional direction and thus a stripe. In the one-dimensional system (1) there is a surprisingly simple criterion concerning the branching of a traveling mode [16]. The bifurcation point can be computed in terms of the stationary DS. In the following we extend this criterion to higher-dimensional systems. However, breathing and traveling are not the only destabilizing modes any more. As the boundary of the excited domain is now a line, there is the possibility of destabilizations, which deform this boundary. Ohta *et al.* [11]

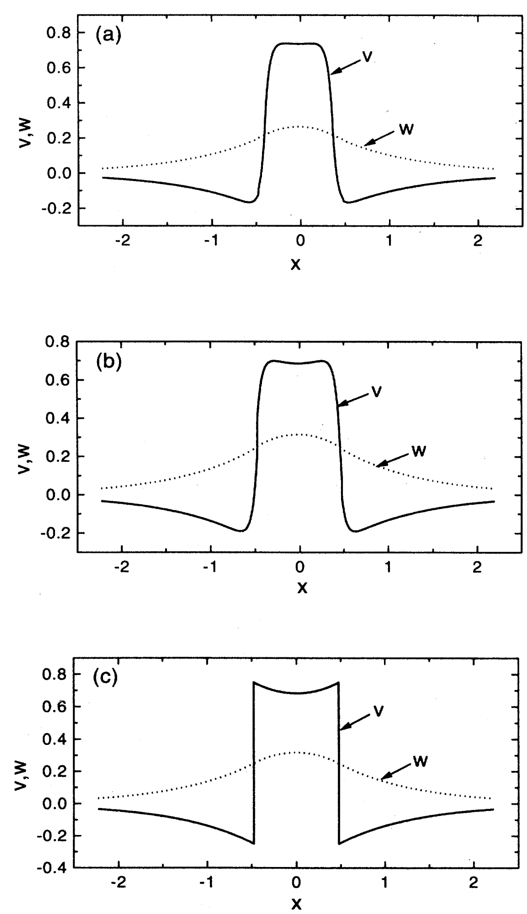

FIG. 2. Typical shape of a dissipative soliton in reaction-diffusion systems of activator inhibitor type depending on ε. When decreasing ε from 0.2 (a) to 0.05 (b), the front-like regions contract. In the singular limit $\varepsilon\to 0$ of (1) they become sharp and the outer domains separate close to the homogeneous stationary state from an excited domain in the middle (c). The patterns are computed for piecewise linear kinetics with $a=0.25$, $\gamma=\frac{1}{3}$.

present the stability analysis of the stationary stripe (or spot) in the case of piecewise linear kinetics and for a reduced dynamics in the range of $\varepsilon \ll 1$. Extending this, we construct the branching traveling stripe and show that subcritical branching is predominant. Studying the traveling pattern, we perform the stability analysis and present the resulting bifurcation set. Finally, we discuss some consequences from the perspective of bifurcation theory.

II. BRANCHING OF A TRAVELING PATTERN

Let us consider an arbitrary stationary solution $(\bar{v}(\mathbf{r}), \bar{w}(\mathbf{r}))$ of the two-dimensional, infinitely extended version of (1) with $\mathbf{r} := (x, y)$. The discussion of a system on a torus is completely analogous. To analyze the stability of that solution, we decompose $(v, w) = (\bar{v}, \bar{w}) + (\tilde{v}, \tilde{w})$ with $(\tilde{v}, \tilde{w})$ being a small deviation:

$$\begin{aligned} \tau\varepsilon \tfrac{\partial}{\partial t}\tilde{v} &= \varepsilon^2 \Delta\tilde{v} + f'(\bar{v})\tilde{v} - \tilde{w}, \\ \tfrac{\partial}{\partial t}\tilde{w} &= \Delta\tilde{w} + \tilde{v} - \gamma\tilde{w}. \end{aligned} \tag{3}$$

Due to the indifference of the pattern toward translations, there is always a set of eigenfunctions with zero eigenvalues. These used to be called Goldstone modes. Each of them causes an infinitesimal shift of the pattern in a certain direction $\mathbf{r_0}$ and thus is proportional to the respective directional derivative $(\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})$, with $\partial_{\mathbf{r_0}} = \frac{\mathbf{r_0}}{|\mathbf{r_0}|} \cdot \nabla$. This can be directly seen when taking an arbitrary directional derivative of the stationary equation (1). We divide the first line of (1) by $\tau\varepsilon$ and get

$$\begin{aligned} 0 &= \begin{pmatrix} \tau^{-1}\varepsilon \;\; \Delta\partial_{\mathbf{r_0}}\bar{v} + (\tau\varepsilon)^{-1} f'(\bar{v})\partial_{\mathbf{r_0}}\bar{v} - (\tau\varepsilon)^{-1}\partial_{\mathbf{r_0}}\bar{w} \\ \Delta\partial_{\mathbf{r_0}}\bar{w} + \partial_{\mathbf{r_0}}\bar{v} - \gamma\partial_{\mathbf{r_0}}\bar{w} \end{pmatrix} \\ &=: L \begin{pmatrix} \partial_{\mathbf{r_0}}\bar{v} \\ \partial_{\mathbf{r_0}}\bar{w} \end{pmatrix}. \end{aligned} \tag{4}$$

How is the bifurcation point representing the branching of traveling DS characterized? Since a DS $(\bar{v}, \bar{w})$ traveling with velocity c into direction $\mathbf{r_0}$ is a solution of

$$\begin{aligned} -c\partial_{\mathbf{r_0}} v &= \tau^{-1}\varepsilon\Delta v + (\tau\varepsilon)^{-1}[f(v, a) - w], \\ -c\partial_{\mathbf{r_0}} w &= \Delta w + v - \gamma w, \end{aligned} \tag{5}$$

the right hand side has to permanently generate the Goldstone mode $g = (\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})$. This might be seen as a manifestation of the ancient Aristotelian concept of propagation as an active process as opposed to the mere force free, momentum controlled motion, which later came to dominate human imagination regarding nature's principles. In the framework of a linear stability analysis of a stationary DS $(\bar{v}, \bar{w})$, the bifurcation to traveling ones thus requires a generator e_g obeying $L\, e_g = g$, leading to a (g, e_g)-subspace representation of L as $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ The existence of such an e_g can be characterized making use of $L^\dagger$ — the adjoint of the operator L — and its zero eigenfunction $g^\dagger$. Abbreviating spatial integration by $\langle\;\rangle$, the key equation results as follows: From $L^\dagger g^\dagger = 0$ we have $0 = \langle L^\dagger g^\dagger | e_g\rangle = \langle g^\dagger | L e_g\rangle$. Thus we find

$$\langle g^\dagger | g\rangle = 0 \tag{6}$$

as a necessary condition. As outlined in [16], system (1) has the exceptional property that $g^\dagger$ can be expressed in terms of $(\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})$ for arbitrary ε. It is $(-\varepsilon\tau\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})$. In general, we cannot assume a symmetrical pattern $(\bar{v}, \bar{w})$. Thus there is a direction $\mathbf{r_0}$ where the above mentioned branching occurs first when decreasing τ. Otherwise, we consider a representative $\mathbf{r}_{\mathrm{repr}}$ of a certain set of directions. We can directly conclude that

$$\begin{aligned} &\langle(-\varepsilon\tau\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})(\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})\rangle \\ &\quad = \langle(-\varepsilon\tau\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w}) L e_g\rangle \\ &\quad = \langle \underbrace{L^\dagger(-\varepsilon\tau\partial_{\mathbf{r_0}}\bar{v}, \partial_{\mathbf{r_0}}\bar{w})}_{0}\, e_g\rangle = 0 \end{aligned} \tag{7}$$

is necessary for e_g to exist. In other words, the critical value of τ for the branching of traveling patterns is

$$\varepsilon\tau = \frac{\int(\partial_{\mathbf{r_0}}\bar{w})^2\, dx\, dy}{\int(\partial_{\mathbf{r_0}}\bar{v})^2\, dx\, dy}. \tag{8}$$

III. CONSTRUCTION OF A TRAVELING STRIPE

Let us return to a stripelike DS. Here the criterion is independant of the direction $\mathbf{r_0}$. Drawing a distinction between different directions is misleading, since it incorporates shifts parallel to the stripe, which are meaningless, of course. In principle, the computation of the traveling pattern is a problem in one-dimensional space. For simplicity, we restrict our considerations to patterns moving in positive directions. In the singular limit $\varepsilon \to 0$ the pulse consists of a front at p_+ and a back at p_-. Both contract to a width of $O(\varepsilon)$; see Fig. 2 for a stationary pattern. Their velocities depend on the local values $\bar{w}(p_\pm)$ at the front position p_+ and the back position p_-. These values are determined by the inhibitor dynamics with v slaved to the respective branch of the characteristic aside from front and back. To be specific, let us study the situation for piecewise linear kinetics. We transform the coordinates to a moving frame $z = x - ct$ with arbitrary but fixed velocity c. Then we look for a distance $d = p_+ - p_-$ such that there is an appropriate solution $\bar{w}$ of the domain dynamics that is stationary in the frame.

In the excited domain between p_- and p_+, component v takes the value $\bar{v}(z) = 1 - \bar{w}(z)$; outside this domain, $\bar{v}(z) = -\bar{w}(z)$ holds. Thus, for $\varepsilon \to 0$ the inhibitor equation reads

$$0 = c\bar{w}'(z) + \bar{w}''(z) + \left\{ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right\} - (1 + \gamma)\bar{w}(z), \tag{9}$$

with 1 between p_- and p_+ and 0 elsewhere. For $z \to \pm\infty$ the value of $\bar{w}$ approaches the homogeneous equilibrium value $w = 0$. The excited domain between p_- and p_+ can be interpreted as a moving inhibitor source. The inhibitor produced diffuses and decays laterally. Due to the linearity of the equation, the effects of different parts of this source can linearly be superposed. The Green's function dependence on c can be obtained from (9)

$$G_0(z) = \frac{1}{2\sqrt{\frac{c^2}{4}+1+\gamma}} \exp\left(-\frac{c}{2}z - |z|\sqrt{\frac{c^2}{4}+1+\gamma}\right). \tag{10}$$

A solution of (9) is stationary in the moving frame. Its values $w(p_\pm)$ just depend on the pulse width $d = p_+ - p_-$:

$$\bar{w}(p_+) = \int_0^d G_0(z)\,dz \quad \text{and} \quad \bar{w}(p_-) = \int_0^d G_0(z-d)\,dz. \tag{11}$$

By explicit construction of the solution, the dependence of the velocities of the front (c_+) [back (c_-)] on the local inhibitor value $w(p_\pm)$ can be expressed as

$$\dot{p}_\pm = c_\pm = \frac{\mp 2[\bar{w}(p_\pm) - (\frac{1}{2} - a)]}{\tau\sqrt{\frac{1}{4} - [\bar{w}(p_\pm) - (\frac{1}{2} - a)]^2}}; \tag{12}$$

compare [12]. Thus we can uniquely determine d as a function of c: For any traveling DS, d has to be chosen numerically such that $c_+(w(p_+)) = c_-(w(p_-))$. Then a matching of $c_\pm$ with the velocity c of the frame occurs for a certain value of τ. Altogether, we can parametrize the whole family of traveling stripes by their velocity c.

The stationary pattern possesses the reflection symmetry

$$\begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix}\left(z - \frac{p_+ + p_-}{2}\right) = \begin{pmatrix} \bar{v} \\ \bar{w} \end{pmatrix}\left(\frac{p_+ + p_-}{2} - z\right). \tag{13}$$

Due to this symmetry, the bifurcation to traveling DSs is a pitchfork bifurcation. The possibility of computing the traveling DSs and the corresponding values of τ as a function of c offers a way to systematically distinguish between supercritical and subcritical branching from the stationary solution. For larger values of ε this has already been analyzed for some selected parameter values [17]. In our case we get a subcritical branching within nearly the whole parameter range. There is only a very small region close to the case of odd symmetrical kinetics ($a = \frac{\gamma}{2(1+\gamma)}$) with supercritical bifurcations. The limit $a \to \frac{\gamma}{2(1+\gamma)}$ leads to a diverging width d of the DS. Then coupling between front and back, which is mediated by the domain variable w, vanishes and thus we consider more and more the case of the bifurcation of a single stationary front. This approach is correct only in the case of an odd symmetrical kinetics as outlined in [25]. Consequently, we find a supercritical branching for a single front in the case of the piecewise linear kinetics, which is outlined in the Appendix. However, this result cannot be generalized at all. Without odd symmetry of the characteristic $f(v,a)$ for fixed parameter a, the expected bifurcation for a single front is transcritical and thus fundamentally different from the pitchfork bifurcation of the DS with arbitrarily large width. Note, however, that for vanishing inhibitor diffusion the bifurcation of the single front is pitchfork irrespective of form and symmetry properties of $f(v,a)$ in (1); see [16].

IV. STABILITY ANALYSIS OF THE TRAVELING STRIPE

In the following, we study the stability of a traveling stripelike DS. In the linear approximation (3) only the Laplace operator acts nontrivially in the lateral spatial direction. Thus the y dependence of the eigenmodes is harmonic, i.e., we can write the corresponding perturbations of front and back lines as

$$p(y) = p_\pm + \tilde{p}_\pm(t)\cos(ky+\phi). \tag{14}$$

The time dependence of the amplitudes $\tilde{p}_\pm \cos(ky+\phi)$ is

$$\frac{d}{dt}\tilde{p}_\pm = \lambda \tilde{p}_\pm, \tag{15}$$

with λ being the corresponding eigenvalue. The dynamics of v in the domains is slaved by w. The corresponding change of $\tilde{v} := v - \bar{v}$ depends on $\tilde{w} := w - \bar{w}$ according to the relation $\tilde{v}(z) = -\tilde{w}(z)$. Close to the front lines, the variation of the inhibitor source is described in terms of a δ function. Altogether, the corresponding eigenmode fulfills

$$\begin{aligned}\lambda\tilde{w} = {} & c\tilde{w}_z + \tilde{w}_{zz} + \tilde{w}_{yy} - (1+\gamma)\tilde{w} \\ & + \delta(z-p_\pm)[\pm\tilde{p}_\pm \cos(ky+\phi)].\end{aligned} \tag{16}$$

$\tilde{w}$ has the form $\hat{w}(z)\cos(ky+\phi)$:

$$\lambda\hat{w} = c\hat{w}_z + \hat{w}_{zz} - (1+\gamma+k^2)\hat{w} + \delta(z-p_\pm)(\pm\tilde{p}_\pm). \tag{17}$$

According to the preceding section, Eq. (17) can be solved with the help of its Green's function, which now additionally depends on the eigenvalue λ. We will restrict our considerations to bifurcation points. That is, $\lambda = i\omega$ for Hopf bifurcation points and $\lambda = 0$ for stationary instabilities in the moving frame. This yields

$$\begin{aligned}G_{k,\omega}(z) = {} & \frac{1}{2\sqrt{\frac{c^2}{4}+1+\gamma+k^2+i\omega}} \\ & \times \exp\left(-\frac{c}{2}z - |z|\sqrt{\frac{c^2}{4}+1+\gamma+k^2+i\omega}\right).\end{aligned} \tag{18}$$

We therefore have to take as a complex square root the one with a positive real part. The values $\hat{w}(p_\pm)$ are then

$$\hat{w}(p_+) = -G_{k,\omega}(d)\tilde{p}_- + G_{k,\omega}(0)\tilde{p}_+, \tag{19}$$

$$\hat{w}(p_-) = -G_{k,\omega}(0)\tilde{p}_- + G_{k,\omega}(-d)\tilde{p}_+. \tag{20}$$

The dynamics of $\tilde{p}$ is based on three sources. First, the local value $\bar{w}$ at $p_\pm + \tilde{p}_\pm \cos(ky+\phi)$ changes due to the displacement $\tilde{p}_\pm$ relative to the moving frame. This change is proportional to the slope of $\bar{w}(z)$ at $p_\pm$. Second, there is the local modification of w due to the dynamics of $\hat{w}$.

These two influences are illustrated in Fig. 3. Third, the velocity depends on the local curvature of front and back lines. The corresponding approximating relation for the velocity in the normal direction has been computed for various problems; see Meron [19]:

$$c_{2dim} = c_{1dim} + \varepsilon \frac{K}{\tau}, \tag{21}$$

with K being the local curvature of the front. Altogether, we get as approximation

$$\lambda \tilde{p}_\pm = c'_\pm(\bar{w}(p_\pm))[\bar{w}'(p_\pm)\tilde{p}_\pm + \hat{w}(p_\pm)] - \varepsilon k^2 \tau^{-1} \tilde{p}_\pm, \tag{22}$$

with $\bar{w}_z(p_+) = G_0(d) - G_0(0)$ and $\bar{w}_z(p_-) = G_0(0) - G_0(-d)$. Note that from (12) the following holds:

$$\begin{aligned} c' &:= c'_+(\bar{w}(p_+)) = -c'_-(\bar{w}(p_-)) \\ &= \frac{-1}{2\tau\{\frac{1}{4} - [\frac{1}{2} - a - \bar{w}(p_+)]^2\}^{\frac{3}{2}}}. \end{aligned} \tag{23}$$

A word is now in order concerning the curvature term. It seems to be inconsequential to regard only this term linear in ε. However, when choosing larger and larger wave numbers k there is an arbitrarily large influence of curvature in (22). For a fixed but arbitrary value of ε this influence dominates and has an important effect. For $k \to \infty$, the term $\hat{w}(p_\pm)$ in (22) vanishes completely because the front modulation, as the driving source in Eq. (16), is of high spatial wave number. Hence, the resulting effect on $\hat{w}$ is smoothed out by the diffusion of w. Thus, when formally omitting $\varepsilon k^2 \tilde{p}_\pm$, the remaining first term leads to an instability at high wave numbers k, and that is physically invalid. Inserting $\hat{w}(p_\pm)$ and $\bar{w}_z(p_\pm)$ into (22), we get

$$\begin{aligned} i\omega \tilde{p}_+ &= c'[G_0(d) - G_0(0)]\tilde{p}_+ + c'[-G_{k,\omega}(d)\tilde{p}_- \\ &\quad + G_{k,\omega}(0)\tilde{p}_+] - \varepsilon k^2 \tau^{-1} \tilde{p}_+, \\ i\omega \tilde{p}_- &= -c'[G_0(0) - G_0(-d)]\tilde{p}_- - c'[-G_{k,\omega}(0)\tilde{p}_- \\ &\quad + G_{k,\omega}(-d)\tilde{p}_+] - \varepsilon k^2 \tau^{-1} \tilde{p}_-, \end{aligned} \tag{24}$$

and, by elimination of $\tilde{p}_+$ and $\tilde{p}_-$, a final critical eigenvalue equation

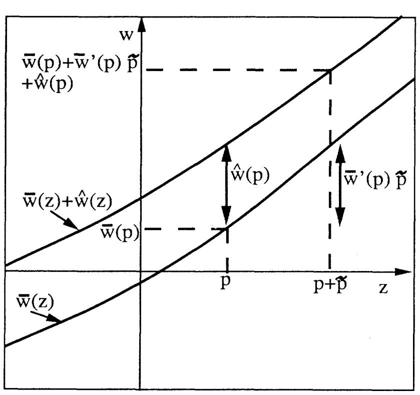

FIG. 3. Local value of $w(p_+)$, which determines the velocity of the front, changes due to the dynamics of w ($\hat{w}$) and the shift of the front position $[\bar{w}'(p_+)\tilde{p}_+]$.

$$\begin{aligned} 0 = &\{-c'[G_0(d) - G_0(0) + G_{k,\omega}(0)] + \varepsilon k^2 \tau^{-1} + i\omega\} \\ &\times\{-c'[G_0(-d) - G_0(0) + G_{k,\omega}(0)] + \varepsilon k^2 \tau^{-1} + i\omega\} \\ &-c'^2 G_{k,\omega}(d) G_{k,\omega}(-d). \end{aligned} \tag{25}$$

Before we start to evaluate this equation for $\varepsilon = 0.05$ and $\gamma = 1/3$, we have to check the limits of our approximation. There are three points. First, ignoring the direct interaction of front and back caused by the activator component v excludes consideration of patterns with small width d. Neglecting this for a moment, we can compute a traveling pattern as discussed above. Decomposing $\bar{v}$ into front and back parts — $\bar{v} = \bar{v}_f(x - p_+) + \bar{v}_b(x - p_-) - \bar{v}_b(\infty)$ — a correction Δc of front and back velocity can be computed using an extension of [26] and [27] to this case. The result for the back, which is always larger than the corresponding expression for the front, is

$$\begin{aligned} \tau \Delta c \approx &\left(-\frac{c\tau}{2} - \sqrt{\frac{(c\tau)^2}{4} + 1}\right) \sqrt{\frac{(c\tau)^2}{4} + 1} \\ &\times \exp\left[-\left(-\frac{c\tau}{2} + \sqrt{\frac{(c\tau)^2}{4} + 1}\right)\frac{d}{\varepsilon}\right]. \end{aligned} \tag{26}$$

This expression becomes large for small patterns, i.e., for a close to $\frac{1}{2}$ and $c \approx 0$. In those situations that we are going to analyze later, $\tau \Delta c$ is at most 0.03 and rapidly decreases by orders of magnitude when changing c or a.

But even for small values of $\tau \Delta c$ there may be deviations of the singular limit dynamics due to imperfect separation of scales for $\varepsilon \not\ll 1$. This is the second point. However, for $c = 0$ and $\varepsilon = 0.05$, Ohta, Mimura, and Kobayashi [11] investigate such deviations of the reduced dynamics from the full dynamics. They find negligibly small differences.

A third point is the influence of curvature. Choosing any fixed perturbation amplitude, large wave numbers k require consideration of higher orders of ε. But these wave numbers are of no relevance for stability. Thus this effect is of no significance in our analysis. In the following, we numerically evaluate Eq. (25) for $\varepsilon = 0.05$ and $\gamma = \frac{1}{3}$. For $c = 0$ our results coincide with those of [11]. There, we study stationary bifurcations and transitions to oscillating front patterns via Hopf bifurcations. The bifurcating modes are of zig-zag type if p_- and p_+ have equal signs. In the case of a varicose mode the signs are different; see Fig. 4. For $c > 0$, there are lines of stationary bifurcation points starting from $c = 0$. With increasing velocity $c > 0$ the mirror symmetry (13) of the stationary pattern gets lost. Thus the absolute values of $\tilde{p}_\pm$ of an eigenmode differ from each other. However, the relation between signs is preserved on those lines and with that (relation) the type of instability. Otherwise, there would be a critical mode with $\tilde{p}_-$ or $\tilde{p}_+$ equal to zero. By inspecting (24), this immediately turns out to be possible only for the Goldstone mode with $\tilde{p}_+ = \tilde{p}_-$ for $k = 0$.

In addition to stationary bifurcations, there is the possibility of Hopf bifurcations. For $c = 0$ we can again distinguish between varicose and zig-zag modes. Their difference may be interpreted as a temporal phase shift be-

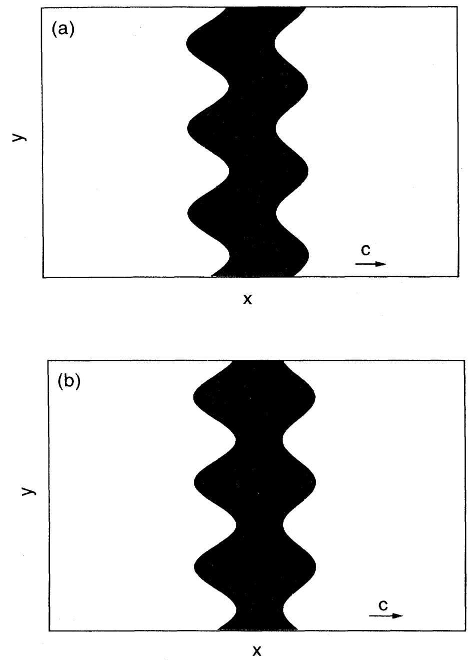

FIG. 4. Typical shape of a zig-zag (a) [varicose (b)] perturbation. The dark domain represents the excited part of the system.

tween $\tilde{p}_+$ and $\tilde{p}_-$, which is π and 0, respectively. But this phase shift is not fixed along the lines of Hopf bifurcation points starting from $c = 0$. Thus, here the above classification cannot be done strictly. However, a line starting at the varicose Hopf instability for $c = 0$ is further called varicose, and the same is done for the case of a zig-zag instability. Let us consider the bifurcation set in typical situations. In the following, we parametrize the family of branching traveling DSs with their velocity c instead of τ, since their dependence on τ is ambiguous due to the subcritical branching; see Fig. 5 for illustration. The bifurcation set is presented in Fig. 6 for different values of a. In a one-dimensional system the pattern becomes stable at a saddle node bifurcation point, which is marked by a dot. It is apparent that the range of stationary

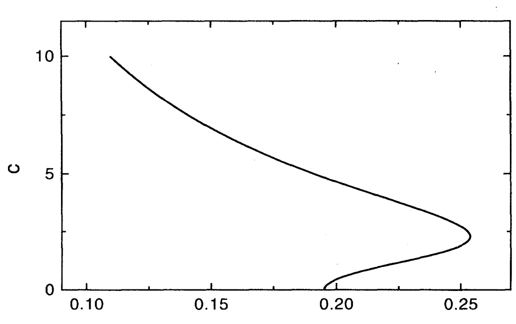

FIG. 5. Relation between velocity c and bifurcation parameter τ. The stationary pattern is destabilized with respect to the traveling mode for decreasing τ. Due to the subcritical branching, the dependency on c (τ) is ambiguous. Parameters are $\gamma = 1/3$ and $a = 0.25$.

varicose instability lies completely within the zig-zag unstable region for all situations considered. Thus it seems to be impossible to get a varicose destabilization of a stable pattern. When $a \to \frac{\gamma}{2(1+\gamma)}$, the width d gets larger and larger. Then coupling between front and back gets weaker and weaker. Consequently, the bifurcation lines of the respective modes converge from Figs. 6(e) to 6(a). In each diagram there is a line of Hopf bifurcation points starting from the varicose bifurcation point at $c = 0$. All these lines terminate below the saddle node bifurcation point. From Figs. 6(a) to 6(e) parameter a is increased. In Fig. 6(a) both stationary instability regions still extend beyond this point, whereas it is only of the zig-zag type in (b). The stationary varicose instability has completely disappeared in Fig. 6(c). Increasing a further, the zig-zag instability sets in at a velocity $c > 0$ and comprises the saddle node point (d). There is a line of Hopf bifurcation points starting from the origin and terminating at the line of static zig-zag instability. At the intersection point we have a codimension-two situation. Due to the symmetry of the pattern, the static bifurcation is of pitchfork type. The surrounding of the codimension-two point represents an unfolding of the complete scenario of a symmetrical Takens-Bogdanov bifurcation; see [28], Chap. 7.3. This implies the existence of additional global bifurcation points. Increasing a further, there eventually are no stationary instabilities at all for wave numbers $k > 0$ (e). Instead, we find a line of zig-zag Hopf bifurcation points joining the origin with the saddle node point. It is this line that an attentive reader may have expected in all the figures. The reason is the following: In the origin and at the saddle node point there is a double zero eigenvalue. Since there is no second zero eigenmode in addition to the Goldstone mode, both degeneracies are of the type $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Arnold [29] presents a miniversal unfolding of this case, which is two dimensional. Within this unfolding, there is a one-dimensional submanifold with purely imaginary eigenvalues. The points on such a manifold represent Hopf bifurcation points in a nonlinear system starting from zero frequency in the degenerate point. However, the submanifold with purely imaginary eigenvalues is limited by the submanifold with zero eigenvalues. At the other side of the latter manifold the eigenvalues are real with different signs. Our unfoldings by c and k are two dimensional and thus we would expect a line of Hopf bifurcation points within them. The submanifold with zero eigenvalue is the line defined by $k = 0$. However, k appears only as a square in (25), i.e., at the SN point we have only one half of the unfolding. At the origin it is just a quarter, since the situation for $c > 0$ is symmetrical to that of $c < 0$. Only when the parameter range close to the SN point is stable with regard to zig-zag perturbations is there a Hopf line. For the unstable case we can find this line for negative k^2, which is of course without physical significance. Referring to the relation between traveling and rocking DS mentioned in the introduction, the incorporation of boundaries also represents a perturbation of the $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ situation due to violation of the translation symmetry. When the distance between DS and the boundaries becomes larger and larger, this perturbation becomes weaker and weaker. Thus the length of the system can be regarded as an additional unfolding parameter. Provided that the boundaries have a repulsive effect on the DS (see, e.g., [1]), there is a stable stationary DS for large values of τ, i.e., we can explicitly exclude the saddle situation. Thus we will find a Hopf bifurcation to swinging DS due to the perturbation.

Let us consider now the last bifurcation bringing stability to the traveling DS with increasing c. In Fig. 6(e) this is a Hopf instability and in (a)–(d) the stationary zig-zag instability. The last modes to stabilize are those close to $k = 0$. Therefore, the zero eigenvalue at $k = 0$, which is due to translation symmetry, remains unaffected. This scenario is known from the Eckhaus instability occurring in hydrodynamics and in connection with the Turing bifurcation in reaction-diffusion systems. There, the primarily branching mode has a zero eigenvalue and the neighboring modes are the last to stabilize. When varying c, the evolution of the eigenvalues is indeed the same in our situation, which can be seen in Fig. 7. The wave number with the largest positive eigenvalue lies in the midst of the unstable region. Finally, we would like to stress that the location of the last stationary bifurcation point depends on the value of ε. This might be aston-

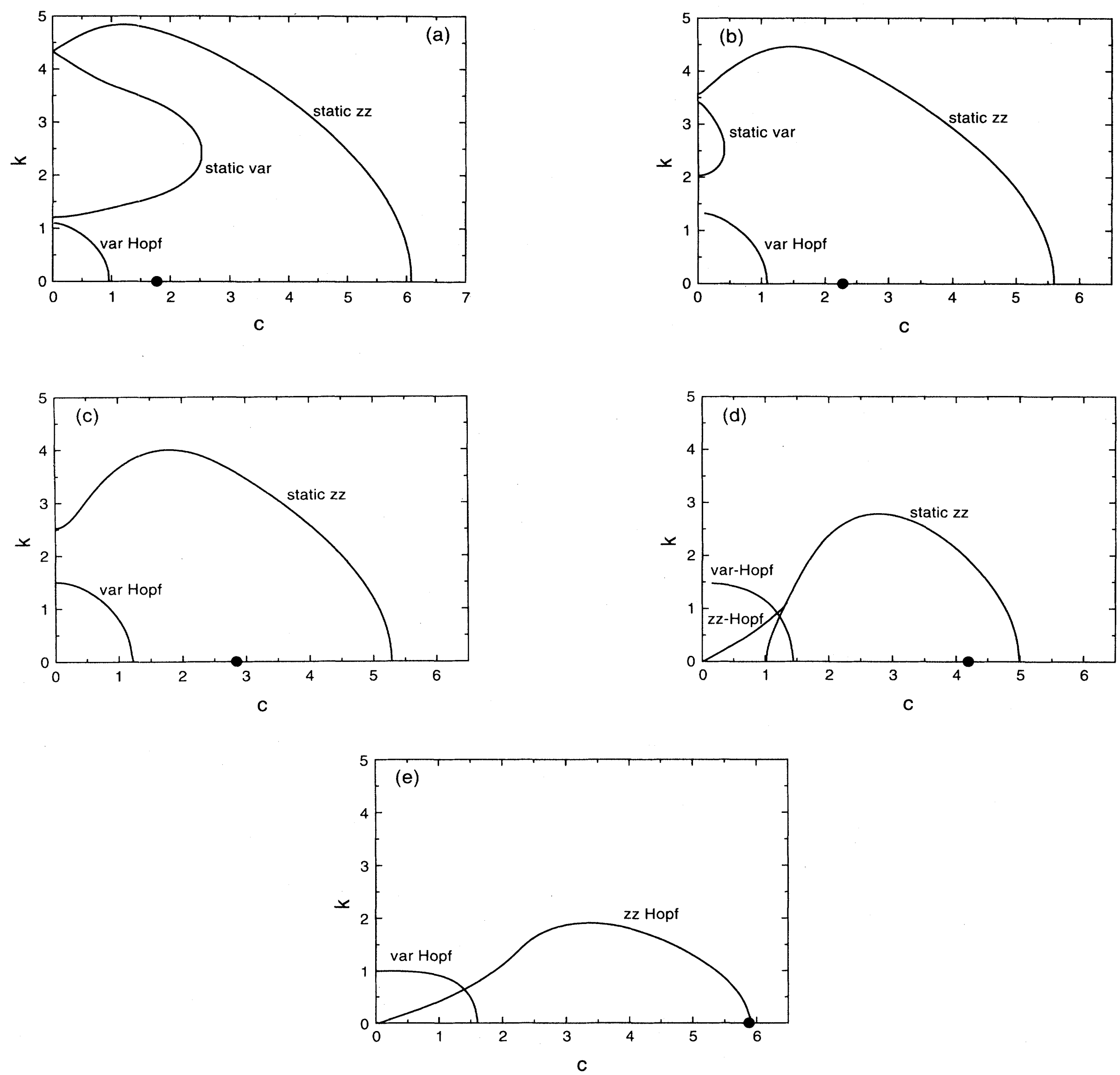

FIG. 6. Dependence of bifurcation points on c and k. Zig-zag and varicose are abbreviated as zz and var, respectively. $\gamma = \frac{1}{3}$. The saddle node bifurcation in a one-dimensional system is marked by a dot. The lines of Hopf bifurcation lines are denoted according to the situation at $c = 0$. The values of a are 0.2 (a), 0.25 (b), 0.3 (c), 0.38 (d), 0.43 (e).

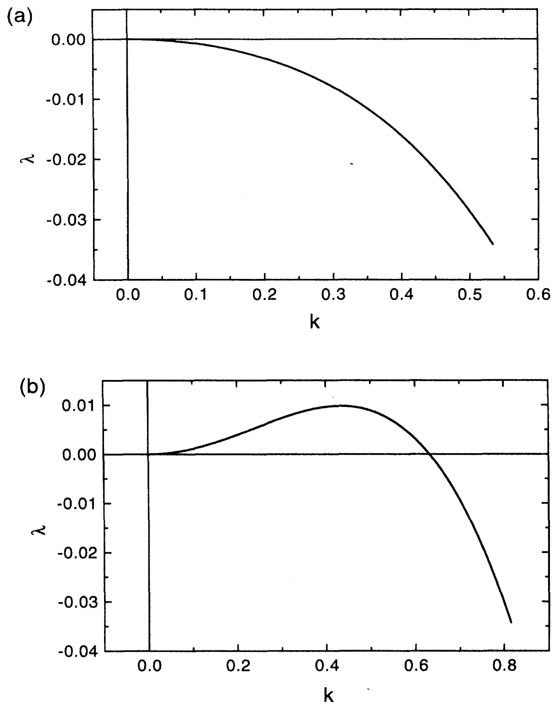

FIG. 7. Dependence of the largest eigenvalue on the wave number k for a stable (a) [unstable (b)] pattern. The parameter values are $a = 0.38$, $\gamma = \frac{1}{3}$. In (a) we have $\tau = 0.0698\,(c = 4.9)$, whereas (b) is computed for $\tau = 0.0692\,(c = 5.05)$.

ishing at first, since this last bifurcation corresponds to $k \to 0$, and for $k = 0$ the influence of ε, entering along with the curvature of the perturbation, vanishes in our approximation. However, emphasis is on $k \to 0$ in the previous sentence. The case $k = 0$ is qualitatively different, since the corresponding perturbation is a mere translation with eigenvalue zero due to translation symmetry, i.e., for $k = 0$, especially in the one-dimensional case, there is no zig-zag instability at all. The limit $k \to 0$ is governed by the second derivative of the spectrum $\lambda(k)|_{k=0}$ (see Fig. 7), which depends on ε.

We evaluated (25) for other values of γ. When changing a, we found no qualitative differences from the presented scenarios.

V. SUMMARY

We analyzed the branching off of traveling DS's from stationary localized patterns in two-dimensional reaction-diffusion media. For systems with Bonhoeffer–van der Pol kinetics, an integral criterion allows for the computation of the bifurcation point in terms of the stationary pattern. Close to the singular limit an analytical treatment of stripelike patterns is possible for piecewise linear kinetics. The bifurcation to traveling patterns appears to be predominantly subcritical. The traveling stripes undergo a sequence of bifurcations, where the last mode to stabilize is typically of zig-zag type. The related bifurcation scenario corresponds to that of the Eckhaus instability. In connection with this result, the question about branching patterns is of great interest. A subcritical bifurcation would possibly lead to a decay of the stripe into a large number of spots, whereas a supercritical bifurcation leads to laterally periodically modulated patterns. The supercritical case could be studied by a weakly nonlinear analysis, whereas the subcritical case essentially requires numerical computations.

ACKNOWLEDGMENT

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APPENDIX: BIFURCATION OF A SINGLE FRONT STRUCTURE

There is a stationary solution of (1) with piecewise linear kinetics (2) with a single front for $a_0 = \frac{\gamma}{2(1+\gamma)}$. Let us treat the case of a front from high to low values of v. In the limit $\varepsilon \to 0$ the velocity of this front depends on the local value of the domain variable $w(p)$:

$$\tau c_v(w(p)) = \frac{2[\frac{1}{2} - a_0 - w(p)]}{\sqrt{\frac{1}{4} - [\frac{1}{2} - a_0 - w(p)]^2}} = \frac{2[\frac{1}{2(1+\gamma)} - w(p)]}{\sqrt{\frac{1}{4} - [\frac{1}{2(1+\gamma)} - w(p)]^2}}. \tag{A1}$$

Depending on this value $w(p)$ we can compute the velocity for the domain variable

$$c_w(w(p)) = \frac{4\sqrt{1+\gamma}[\frac{1}{2(1+\gamma)} - w(p)]}{\sqrt{\frac{1}{1+\gamma} - 4[w(p) - \frac{1}{2(1+\gamma)}]^2}}, \tag{A2}$$

that is, the velocity of a solution of (9) with 1 for $z < p$ and 0 for $z > p$ and $w = w(p)$ at the front position p. For $w_0 := w(p) = \frac{1}{2(1+\gamma)}$ there is a stationary front independent of τ. For a bifurcating branch of front with $c \neq 0$ to exist, we must require that $c_v(w(p)) = c_w(w(p))$. Decreasing τ we get destabilization of the stationary front when passing the bifurcation point with $c_v'(w_0) = c_w'(w_0)$. This condition is fulfilled for $\tau_c = \frac{1}{1+\gamma}$. Due to the odd symmetry of the kinetics, the second derivatives are zero at that point. Thus there is a pitchfork bifurcation, and the third derivatives for $\tau = \tau_c$ determine whether a subcritical or a supercritical bifurcation occurs. We compute $c_w''' = (1+\gamma)c_v'''$. For positive γ, branching solutions occur for decreasing τ and thus we have a supercritical bifurcation. In general, the kinetics has no odd symmetry. The second order terms at the zeros of the equations corresponding to (A1) and (A2) generically result in a transcritical bifurcation. To study the pitchfork bifurcation of a DS we must consider the coupling between front and back in any case.

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